Due Sun

1.6 – More on Linear Systems and Invertible Matrices

Theorem 1.6.1 A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

We know $A\vec{x}=\vec{b}$ can be inconsistent or can have exactly one solution

If $A\vec{x}=\vec{b}$ has solutions $\vec{x}_1$ & $\vec{x}_2$, then

Let $\vec{x}_0=\vec{x}_1-\vec{x}_2$. Then $A\vec{x}_0=A(\vec{x}_1-\vec{x}_2)$

$$=A\vec{x}_1-A\vec{x}_2=\vec{b}-\vec{b}$$

$$\Rightarrow A\vec{x}_0=\vec{0}$$

If k is any scalar, then

$A(\vec{x}_1+k\vec{x}_0)=A\vec{x}_1+kA\vec{x}_0=\vec{b}$. Infinitely many choices for k $\Rightarrow$ infinitely many solutions exist.

Theorem 1.6.2 If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$. because A^{-1} is unique.

4. Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + x_3 = 5$$

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to 3R_1 - 5R_2$

(Not elementary but allowed)

$$A^{-1}=\begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}$$

(verify when you make time)

$$\vec{x} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

11. Solve the linear systems. Using the given values for the b's solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

Invertible if $4(2) - 1(-7) \neq 0$ ✓

i. $b_1 = 0, b_2 = 1$
ii. $b_1 = -4, b_2 = 6$
iii. $b_1 = -1, b_2 = 3$
iv. $b_1 = -5, b_2 = 1$

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 1 & 2 & b_2 \end{array}\right]$$

$R_2 \Rightarrow R_1 - 4R_2$

$$\begin{array}{ccc} 4 & -7 & b_1 \\ -4 & -8 & -4b_2 \\ \hline 0 & -15 & b_1 - 4b_2 \end{array}$$

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 0 & -15 & b_1 - 4b_2 \end{array}\right]$$

$R_1 \Rightarrow 15R_1 - 7R_2$

$$\begin{array}{ccc} 60 & -105 & 15b_1 \\ 0 & 105 & -7b_1 + 28b_2 \\ \hline 60 & 0 & 8b_1 + 28b_2 \end{array}$$

$$\left[\begin{array}{cc|c} 60 & 0 & 8b_1 + 28b_2 \\ 0 & -15 & b_1 - 4b_2 \end{array}\right]$$

$R_1 \Rightarrow \frac{1}{60} R_1$
$R_2 \Rightarrow -\frac{1}{15} R_2$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2}{15}b_1 + \frac{7}{15}b_2 \\ 0 & 1 & -\frac{1}{15}b_1 + \frac{4}{15}b_2 \end{array}\right]$$

ii) $(b_1, b_2) = (-4, 6)$

ii)
$(x_1, x_2) = \left(\frac{34}{15}, \frac{28}{15}\right)$

i) $\left(\frac{7}{15}, \frac{4}{15}\right)$

iii) $\left(\frac{19}{15}, \frac{13}{15}\right)$

iv) $\left(-\frac{1}{5}, \frac{3}{5}\right)$

17. Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

rref: $\left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$

The system is consistent if

$$\begin{aligned} b_1-b_2+b_3 &= 0 \\ -2b_1+b_2+b_4 &= 0 \end{aligned} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}$$

rref: $\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & -1 \end{bmatrix}$

The system is consistent if $\boxed{\begin{aligned} b_1 &= b_3+b_4 \\ b_2 &= 2b_3+b_4 \end{aligned}}$ and

Theorem 1.6.4 Equivalent Statements (extends Theorem 1.5.3)

If A is an $n \times n$ matrix, then the following are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Theorem 1.6.5 Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Note: We already saw that if A & B are invertible, then AB is invertible.

Pf: Let $\vec{x}_0$ be a solution of $B\vec{x} = \vec{0}$.

Then $(AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}$

$\vec{x}_0 = \vec{0}$ by thm 1.6.4 a & b, applied to the invertible matrix AB.

Then $B\vec{x} = \vec{0}$ has only the trivial solution $\Rightarrow$ B is invertible. ✓

$$A = A(BB^{-1}) = \underbrace{(AB)}_{\text{invertible}}\underbrace{B^{-1}}_{\text{invertible}}$$

so A is invertible. ✓